Resampled random processes in gravitational-wave data analysis

A. Królak

Institute of Mathemathics, Polish Academy of Sciences, Śniadeckich 8, 00-950 Warsaw, Poland

Massimo Tinto

Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California 91109.

Abstract

To detect a continuous gravitational-wave signal it is necessary to take into account the motion of the detector with respect to the solar system barycenter. In order to search efficiently for such signals by means of the fast Fourier transform one needs to transform the data from the topocentric time to the barycentric time by means of resampling. The resampled data form a non-stationary random process. In this communication we prove that this non-stationary random process can be mathematically well defined and we show that generalizations of the fundamental results for stationary processes like Wiener-Khintchine theorem and Cramèr representation exist.

1   Introduction

Continuous gravitational-wave signals originating for example from the spinning neutron stars are believed to be very weak. In order to detect them in the data of both bar and laser interferometric gravitational-wave detectors one needs to integrate the data for many days. In such a case in the model of the signal one needs to take into account motion of the detector with respect to the solar system barycenter (SSB).

The simplest model of such a signal for the case of monochromatic gravitational-wave source has the form

$$s = A\cos[\phi_0 + 2\pi f_0 t + \frac{2\pi}{c}f_0 \mathbf{n}_0 \cdot \mathbf{r}_{\text{SSB}}(t)] \qquad (1)$$

In Eq. (1) ϕ_0 is the initial phase of the waveform, $\mathbf{r}_{\text{SSB}}$ is the vector joining the solar system barycenter (SSB) with the detector, $\mathbf{n}_0$ is the constant unit

vector in the direction from the SSB to the neutron star. We assume that the gravitational wave form is monochromatic with frequency f_0 which we define as instantaneous frequency evaluated at the SSB at $t = 0$. In general frequency of the signal will change for example due to spindown of the rotating neutron star but this this not introduce new qualitative features into the mathematical model. The above signal cannot be detected by means of the fast Fourier transform algorithm (FFT) because the unknown frequency f_o to be estimated is contained in the highly non-linear function of time depending on the position of the detector with respect of the SSB. Consequently the following technique is proposed[1]. Introduce a new time t_B called barycentric time which is related to the topocentric time t by

$$t_B = t + \mathbf{n}_0 \cdot \mathbf{r}_{\text{SSB}}(t)/c. \tag{2}$$

Then the signal given above takes the form

$$s_B = A \cos(\phi_0 + 2\pi f_o t_B). \tag{3}$$

The signal s_B is monochromatic and can be searched for by means of FFT. However as we shall show in the following section the transformation given by Eq. 2 make the noise in the data a non-stationary random process. The main result of this communication proven in the next section will be this non-sationary random process can be mathematically well defined and generalizations of the fundamental results for stationary processes like Wiener-Khintchine theorem and Cramèr representation exist. We also that if the data are white noise the resampled process remains stationary.

Another case when the resampling of the data is applied are the so-called "accelarated searches" in pulsar data analysis[2, 3]. These are searches for pulsars in binary systems where observation time is much shorter than the period of the orbit so that we can expand the phase of the signal in Taylor series keeping only the second order terms. This results in the following signal

$$s_a = A \cos[\phi_0 + 2\pi(f_o t + p_o t^2)]. \tag{4}$$

This signal can be detected by matched filtering i.e. by multiplication of the data by $exp(-2\pi p t^2)$ for a grid of prameters p followed by FFT. Nevertheless pulsar astronomers apply resampling also in this case. One defines a new time

$$t_a = t + at^2, \tag{5}$$

where $a = p_o/f_o$. Then the signal s_a becomes monochromatic:

$$s_a = A \cos[\phi_o + 2\pi f_o t_a] \tag{6}$$

and can be searched for by means of FFT. Of course we do not know the frequency f_o and parameter p_o and we have to resample the data for an appropriate grid of acceleration parameter a.

2 Resampled random processes

Let $x(t)$ be a continuous parameter real-valued random process. The process $x(t)$ is said to be *completely stationary*[4] (sometimes referred to as *strongly stationary* or *strictly stationary*) if for all $n \geq 1$, for any $t_1, t_2, ..., t_n$ contained in the index set, and for any τ such that $t_1 + \tau, t_2 + \tau, ..., t_n + \tau$ are also contained in the index set, the joint cumulative probability distribution function(cpdf) of $x(t_1), x(t_2), ..., x(t_n)$ is the same as that of $x(t_1 + \tau), x(t_2 + \tau), ..., x(t_n + \tau)$. In other words, the probabilistic structure of a completely stationary process is invariant under a shift of time. The process $x(t)$ is said to be *second-order stationary*[4] (sometimes referred to as *weakly stationary* or *covariance stationary*) if for all $n \geq 1$, for any $t_1, t_2, ..., t_n$ contained in the index set, and for any τ such that $t_1 + \tau, t_2 + \tau, ..., t_n + \tau$ are also contained in the index set, the joint moments of orders 1 and 2 of $x(t_1), x(t_2), ..., x(t_n)$ exist, are finite and are equal to corresponding joint moments of $x(t_1 + \tau), x(t_2 + \tau), ..., x(t_n + \tau)$. A weakly stationary Gaussian random process is also completely stationary because a Gaussian process is completely determined by its 1st and 2nd moment. For stationary random processes we have the following fundamental result.[5]

Theorem 1 (The Wiener-Khintchine Theorem) *A necessary and sufficient condition for $\rho(\tau)$ to be the autocorrelation function of some stochastically continuous (i.e. continuous in the mean squre sense) stationary process, $x(t)$, is that there exists a function, $F(\omega)$, having the properties of a distribution function on $(-\infty, \infty)$, (i.e. $F(-\infty) = 0, F(\infty) = 1$, and $F(\omega)$ non-decreasing), such that, for all $\tau, \rho(\tau)$ may be expressed in the form,*

$$\rho(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} dF(\omega). \tag{7}$$

The necessary part of the above Theorem follows from a general theorem due to Bochner that any positive semi-definite function which is continuous everywhere must have the representation of the above form. In the case of a purely continuous spectrum we have $dF(\omega) = S(\omega)d\omega$, where $S(\omega)$ is (normalized) spectral density function. Thus the Wiener-Khintchine theorem asserts that a well defined spectral density existsts.

Another important results is the existence of spectral decomposition of the stationary random process itself.

Theorem 2 (The Cramèr representation) *Let $x(t), -\infty < t < \infty$, be a zero-mean stochastically continuous stationary process. Then there exists an orthogonal process, $Z(\omega)$, such that for all t, $x(t)$ may be written in the form,*

$$x(t) = \int_{-\infty}^{\infty} e^{i\omega t} dZ(\omega), \tag{8}$$

the integral being defined in the mean-square sense. The process $Z(\omega)$ has the following properties;

(i) $E[dZ(\omega)] = 0$, for all ω

(ii) $E[|dZ(\omega)|^2] = dH(\omega)$, for all ω,

where $H(\omega)$ is the (non-normalized) integrated spectrum of $x(t)$,

(iii) for any two distinct frequencies, ω , ω', $(\omega \neq \omega')$,

$$cov[dZ(\omega), dZ(\omega')] = E[dZ^*(\omega)dZ(\omega')] = 0. \tag{9}$$

The above result says that any stationary random process can be decomposed into a sum of sine and cosine functions with uncorrelated coefficients.

Let $x(t)$ be a random process and let $t_r = t + k(t; \theta)$ be a smooth one-to-one function both of the index t and the parameter set θ A *resampled random process* $y(t_r)$ is defined as the random process $x(t)$ taken at time t_r i.e. $y(t_r) = x(t)$. In the following for simplicity we assume that the process $x(t)$ is zero-mean. It immediately follows that the resampled process $y(t_r)$ is also zero mean. Suppose that the original process is stationary. Let us first convince ourselves that the resampled process is, in general, *non-stationary*. Let $C(t_r', t_r) := E[y(t_r')y(t_r)]$ be the autocovariance function of the resampled process. By definition of the resampled process we have that $C(t_r', t_r) := E[x(t')x(t)]$ and by stationarity of $x(t)$ we have $C(t_r', t_r) = R(t' - t)$. By implicit function theorem we have that there exists a smooth function $t = t_r + g(t_r; \theta)$. In order that the resampled process be stationary the function R must depend only on the difference $(t_r' - t_r)$ This is the case if and only if t is a linear function of t_r i.e. $t = t_r + a(\theta)t_r + b(\theta)$. Thus when the resampling transformation is non-linear the resulting resampled process is non-stationary.

In the linear resampling case the Fourier transform $\widetilde{Y}(\omega_r)$ of the resampled process at frequency ω_r is related to the Fourier transform $\widetilde{X}(\omega)$ of the original process at frequency ω by the following formula:

$$\widetilde{Y}(\omega_r) = \frac{\exp i\omega b}{1 + a} \widetilde{X}(\omega), \tag{10}$$

where $\omega = \frac{\omega_r}{1+a}$

Let us consider the covariance function $C(t_r', t_r)$, of the resampled process $y(t_r)$. It can be written as

$$C(t_r', t_r) = \int_{-\infty}^{\infty} \phi_{t_r'}^*(\omega)\phi_{t_r}(\omega)dH(\omega), \tag{11}$$

where we have introduced a set functions

$$\phi_{t_r}(\omega) = \exp[i\omega(t_r + g(t_r))] \tag{12}$$

We have the following important result.

Theorem 3 (General orthogonal expansions) *Let $y(t_r)$ be a continuous parameter zero mean process (not necessarily stationary) with covariance function $C(t_r, s_r) = E[y(t_r)y(s_r)]$. If there exists a family of functions, $\{\phi_{t_r}(\omega)\}$, defined on the real line, and indexed by suffix t_r, and a measure, $\mu(\omega)$, on the real*

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line such that for each t, $\phi_{t_r}(\omega)$ is quadratically integrable with respect to the measure μ, i.e.

$$\int_{-\infty}^{\infty} |\phi_{t_r}(\omega)|^2 \, d\mu(\omega) < \infty \tag{13}$$

and for all t_r, s_r, $C(t_r, s_r)$ admits a representation of the form,

$$C(t_r', t_r) = \int_{-\infty}^{\infty} \phi_{t_r'}^*(\omega)\phi_{t_r}(\omega)dH(\omega) \tag{14}$$

then the process $y(t_r)$ admits a representation of the form

$$y(t_r) = \int_{-\infty}^{\infty} \phi_{t_r}(\omega)dZ(\omega), \tag{15}$$

where $Z(\omega)$ is an orthogonal process with

$$E[|dZ(\omega)|^2] = d\mu(\omega). \tag{16}$$

Conversely if $y(t_r)$ admits a representation of the form with an orthogonal process satisfying , then $C(t_r', t_r)$ admits a representation of the form .

The formula (14) is called *generalized Wiener-Khintchine* relation and formula (15) is called *generalized Cramèr representation* of the random process. In our case the generalized Cramèr representaion reads.

$$y(t_r) = \int_{-\infty}^{\infty} \exp[i\omega(t_r + g(t_r))]dZ(\omega), \tag{17}$$

This representation also clearly shows that the resampled process is in general non-stationary because the choice of basis function

$$\phi_{t_r}(\omega) = \exp(i\omega t_r) \tag{18}$$

is not in general possible. The generalized Cramèr representaion is also immediate from the Cramèr representation for the original stationary process and its transformation to resampled time. It also follows that the measure μ coincides with the integarted spectrum $H(\omega)$ of the original stationary process. However $H(\omega)$ cannot be interepreted as the spectrum of the resampled process. Indeed for the resampled process which is non-stationary the concept of spectrum is not mathematically well defined.

The general orthogonal expansion theorem has already been used by M. B. Priestley [5] to develop the theory of so called *evolutionary spectra*. This theory describes a very important class of non-stationary processes often occurring in practice for which the amplitude of the Fourier transform slowly changes with time.

In the case of continuous spectrum we have $dH(\omega) = S(\omega) d\omega$. Then one can write the properties (ii) and (iii) of Cramèr representation theorem as

$$E[\widetilde{X}(\omega')^*\widetilde{X}(\omega)] = S(\omega)\delta\left(\omega' - \omega\right) \tag{19}$$

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where δ is the Dirac function. In the continuous case it is instructive to calculate the correlation function for the Fourier frequency components of the resampled (non-stationary) process. Using Eq.(15) the Fourier transform $\widetilde{Y}(\omega)$ of the resampled process $y(t_r)$ can be written as

$$\widetilde{Y}(\omega) = \int_{-\infty}^{\infty} Q(\omega_1, \omega) dZ(\omega_1), \tag{20}$$

where kernel Q is given by

$$Q(\omega_1, \omega) = \int_{-\infty}^{\infty} \phi_{t_r}(\omega_1) \exp(-i\omega t_r) dt_r. \tag{21}$$

The correlation between two Fourier components of the resampled process takes the form

$$
\begin{aligned}
E[\widetilde{Y}(\omega')^*\widetilde{Y}(\omega)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(\omega_1, \omega')^* Q(\omega_2, \omega) E[dZ(\omega_1)^* dZ(\omega_2)] \\
&= \int_{-\infty}^{\infty} Q(\omega_1, \omega')^* Q(\omega_1, \omega) S(\omega_1) d\omega_1
\end{aligned}
\tag{22}
$$

Thus we see that for a resampled random process the Fourier components at different frequencies are correlated, This is another manifestation of the nonstationarity of the process. Let us now consider the example of white noise for which the spectral density is independent of frequency ω. It is straightforward to show that

$$\int_{-\infty}^{\infty} Q(\omega_1, \omega')^* Q(\omega_1, \omega) d\omega_1 = \delta(\omega' - \omega) \tag{23}$$

Thus for the case of white noise we have that

$$E[\widetilde{Y}(\omega')^*\widetilde{Y}(\omega)] = S(\omega)\delta\left(\omega' - \omega\right) \tag{24}$$

and consequently in this case the noise remains stationary after resampling. It is also easy to see that the Fourier components at different frequencies will be uncorrelated if the spectral density is constant over the bandwidth of the kernel Q. It is possible that the last assumption will be fulfilled in the case of search of the data from gravitational-wave detectors from continuous sources[6] but is not guaranteed.

Acknowledgments

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[6] See discussion in Section IV of Ref. [1].